MATH 1C PRACTICAL SPRING 2023 RECITATION 9

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Logistical note: please remember to fill out the TQFR survey for this class.

1. Fundamental Theorem of Calculus in 2 and 3 Dimensions

The fundamental theorem of calculus in a single variable has two parts. First is the existence of antiderivatives: given $f : [a, b] \to \mathbb{R}$ continuous, there exists a differentiable function $F : [a, b] \to \mathbb{R}$ (defined by $F(x) = \int_a^x f(t)dt$) such that $F'(x) = f(x)$ for all x. The second is about integrating the derivative of a function: if $F'(x) = f(x)$, then $\int_x^y f(t)dt = F(y) - F(x)$.

The analog to the first part in higher dimensions is discussed briefly in Chapter 8.3. For the second part, we need to determine the correct notion of the derivative. This can be the gradient, divergence, or curl, depending on the situation. Furthermore, we interpret the difference $F(y) - F(x)$ as being the integral of the function F over the 0-dimensional set $\{x, y\}$, which is the boundary of the interval $[x, y]$.

In general, there are theorems which state that the integral of some kind of derivative of a function F over a domain D equals the integral over the boundary of the domain ∂D of the function. In the following, we assume our functions are continuously differentiable and our domains are sufficiently nice.

Theorem 1.1 (Fundamental theorem of line integrals). *If $c : [a, b] \to \mathbb{R}^n$ is a C^1 curve and $f : \mathbb{R}^n \to \mathbb{R}$ is a differentiable function, then*

$$\int_C \nabla f \cdot ds = f(c(b)) - f(c(a)).$$

For a domain $D \subset \mathbb{R}^2$, we orient the boundary of D such that travelling around the curve always keeps the interior of D to the left. (Draw an annulus and the orientation of its boundary)

Theorem 1.2 (Green's Theorem). *Let $F : \mathbb{R}^2 \to \mathbb{R}^2$, $D \subset \mathbb{R}^2$, then*

$$\int_{\partial D} F \cdot ds = \iint_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} dxdy.$$

For a surface $S \subset \mathbb{R}^3$, a parametrization $\phi : D \to \mathbb{R}^3$ of S gives an orientation of S by the normal vector. Choose a parametrization $c : [a, b] \to \mathbb{R}^2$ of ∂D with the correct orientation, then this gives a parametrization of ∂S by $\widetilde{c} = \phi \circ c : [a, b] \to \mathbb{R}^3$, which also endows ∂S with an orientation.

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Theorem 1.3 (Stokes' Theorem)**.** *Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field, and $S \subset \mathbb{R}^3$ a surface. Then*

$$\int_{\partial S} F \cdot ds = \iint_S \nabla \times F \cdot dS$$

Note that Green's theorem is a special case of Stokes' theorem. As another special case, if S has no boundary (e.g. a sphere), then the integral over S of the curl of any vector field is zero.

For a solid region in $\mathbb{R}^3$, its boundary is a surface. We orient the boundary with outward pointing normal.

Theorem 1.4 (Gauss' Theorem)**.** *For $F : \mathbb{R}^3 \to \mathbb{R}^3$ a vector field, $W \subset \mathbb{R}^3$ a solid region,*

$$\iint_S F \cdot dS = \iiint_W \nabla \cdot F dxdydz.$$

Exercise 1. Let $F = (2yz, -x + 3y + 2, x^2 + z)$, evaluate $\iint_S \nabla \times F \cdot dS$ where S is the cylinder $x^2 + y^2 = a$, $0 \leq z \leq 1$. What happens if we include the top and bottom of the cylinder?

Exercise 2. Verify Green's theorem for the line integral $\int_C x^2 y dx + y dy$ when C is the boundary of the region between the curves $y = x$ and $y = x^3$, $0 \leq x \leq 1$.

Exercise 3. Evaluate $\int_C x^3 dy - y^3 dx$ where C is the unit circle.

Exercise 4. Evaluate the integral $\iint_S F \cdot dS$ where $F = (x, y, 3)$ and where S is the surface of the unit sphere.

Exercise 5. Let S be the boundary of the cube $[-1, 1]^3 \subset \mathbb{R}^3$, oriented with outward pointing normal. Let $F(x, y, z) = (x + e^y, y + \log x, z + \cos xy)$. Find $\iint_S F \cdot dS$.