MATH 1C PRACTICAL SPRING 2023 RECITATION 8

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1. Line Integrals

We know how to integrate functions over domains which are flat and are bounded by equations. However, in general, we may have a function/vector field defined over a path or surface in $\mathbb{R}^3$. To integrate such functions, we need to parametrize the path/surface in order to reduce the problem to computing a 1 or 2 variable integral. The integral should be independent of the choice of parametrization.

A parametrization of a path is a C^1 function $c : [a, b] \to \mathbb{R}^n$. We require that $c'(t) \neq 0$ for all t; this is so that the path doesn't reverse direction. Let C denote the image of the path, and we give it an "orientation", which is the direction traversed by $c(t)$ as t increases from a to b. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a scalar function, then the path integral of f along C is

$$\int_C f ds = \int_a^b f(c(t))\|c'(t)\|dt.$$

This does not depend on choice of parametrization for C, i.e. if $c_1 : [a_1, b_1] \to \mathbb{R}^n$ is another path which also has image C (with the same orientation), then

$$\int_a^b f(c(t))\|c'(t)\|dt = \int_{a_1}^{b_1} f(c_1(t))\|c_1'(t)\|dt$$

(can prove this using the chain rule). As an example, when $f = 1$ the constant function 1, then $\int_C f ds$ equals the arclength of C.

Now if $F : \mathbb{R}^n \to \mathbb{R}^n$ is a vector field, the path integral of F along C is

$$\int_C F \cdot ds = \int_a^b F(c(t)) \cdot c'(t)dt.$$

This also does not depend on choice of parametrization (again by the chain rule). The dot product measures the amount of the vector field that points in the same direction as the path. For example, if F is a force field in space, then the work done by F on a particle moving along a path c is equal to the path integral of F along c.

Theorem 1.1 (Fundamental theorem of line integrals)**.** *If c is as above and $f : \mathbb{R}^n \to \mathbb{R}$ is a differentiable function, then*

$$\int_C \nabla f \cdot ds = f(c(b)) - f(c(a)).$$

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Exercise 1. Let $c : [0,1] \to \mathbb{R}^2$ be $c(t) = (\cos 2\pi t, \sin 2\pi t)$, compute the path integrals of

(a) $g(x, y) = x + y$
(b) $F(x, y) = 2(x, y)$.

Exercise 2. Consider the gravitational force field (with $G = m = M = 1$) defined by

$$F(x, y, z) = -\frac{1}{(x^2 + y^2 + z^2)^{3/2}}(xi + yj + zk).$$

Show that the work done by the gravitational force as a particle moves from (x_1, y_1, z_1) to (x_2, y_2, z_2) along any path depends only on the radii $R_1 = \|(x_1, y_1, z_1)\|$ and $R_2 = \|(x_2, y_2, z_2)\|$.

2. Surface Integrals

A parametrization of a surface is a differentiable function $\phi : D \to \mathbb{R}^3$, where $D \subset \mathbb{R}^2$. Let S denote the image $\phi(D)$. Assume that the rank of $\operatorname{Jac}\phi$ is equal to 2 at every point. The tangent vectors to the tangent plane of S at $\phi(u, v)$ are

$$\phi_u = \frac{\partial \phi}{\partial u} = \left(\frac{\partial \phi_1}{\partial u}, \frac{\partial \phi_2}{\partial u}, \frac{\partial \phi_3}{\partial u}\right)\Big|_{(u,v)}$$

$$\phi_v = \frac{\partial \phi}{\partial v} = \left(\frac{\partial \phi_1}{\partial v}, \frac{\partial \phi_2}{\partial v}, \frac{\partial \phi_3}{\partial v}\right)\Big|_{(u,v)}.$$

By assumption, these are linearly independent everywhere. The normal vector of S at $\phi(u, v)$ is $n = \phi_u \times \phi_v$, and this is nonvanishing (note that this only makes sense for surfaces in $\mathbb{R}^3$). Note that the vector $-n$ is also normal to the tangent plane; the specific choice of n gives an orientation of the surface.

Exercise 3. The upper unit hemisphere in $\mathbb{R}^3$ can be parametrized by

$$\phi(u, v) = (u, v, \sqrt{1 - u^2 - v^2})$$

for (u, v) in the unit disk. Draw the tangent vectors and normal vector at $\phi(u, v)$ for $(u, v) = (0, 0)$. What happens if you swap the roles of u and v?

Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a scalar function and $F : \mathbb{R}^3 \to \mathbb{R}^3$ a vector field. The surface integral of f on S is

$$\iint_S f dS = \iint_D f(\phi(u, v))\|n(u, v)\| du dv.$$

The surface integral of F on S is

$$\iint_S F \cdot dS = \iint_D F(\phi(u, v)) \cdot n(u, v) du dv.$$

These do not depend on the choice of parametrization. If $f = 1$, then the surface integral of f on S is the surface area of S.

Exercise 4. Calculate $\iint_S F \cdot dS$, where $F(x, y, z) = (x, y, -y)$ and S is the cylindrical surface defined by $x^2 + y^2 = 1$, $0 \le z \le 1$, with normal pointing out of the cylinder.

Exercise 5. Given an electric point charge of q Coulombs placed at a point x_0, the electric field due to the charge q is given by

$$E(x) = \frac{1}{4\pi\epsilon_0}\frac{q}{(x - x_0)^2}\hat{r},$$

where $\hat{r}$ is the unit vector in the direction from x_0 to x. If S is a surface in $\mathbb{R}^3$, the electric flux of E over S is given by

$$\Phi_E = \iint_S E \cdot dS.$$

Let the charge q be placed at the origin, let S be the unit sphere oriented with normal vector pointing outward, and show that

$$\Phi_E = \frac{q}{\epsilon_0}.$$

Remark. In fact, the electric flux for any closed surface S containing the charge is the same, equal to q/ϵ_0.