MATH 1C PRACTICAL SPRING 2023 RECITATION 7

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1. Double and Triple Integrals

Tricks for evaluating double and triple integrals:

- Draw the domain of integration.
- Change the order of integration.
- Do a change of variables.

Definition 1.1. Let $T : D^* \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 transformation given by $x = x(u, v)$ and $y = y(u, v)$. The Jacobian determinant of T, written $|\partial(x, y)/\partial(u, v)|$, is the determinant of the Jacobian matrix $DT(u, v)$ of T:

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}.$$

Theorem 1.2 (Change of variables for double integrals)**.** *Let D, D^* be elementary regions in the plane and let $T : D^* \to D$ be of class C^1 and bijective. Then for any integrable function $f : D \to \mathbb{R}$, we have*

$$\iint_D f(x,y)dxdy = \iint_{D^*} f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv.$$

Example 1. Polar coordinates on $\mathbb{R}^2$ are given by $x = r\cos\theta$, $y = r\sin\theta$. Therefore, the transformation T given by $x = x(r, \theta)$, $y = y(r, \theta)$ has Jacobian determinant

$$\begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$

We can express this fact as $dxdy = rdrd\theta$, and so by the change of variables formula,

$$\iint_D f(x,y)dxdy = \iint_{D^*} f(r\cos\theta, r\sin\theta) r dr d\theta.$$

Exercise 1. Evaluate the following integrals.

(1)

$$\int_0^3 \int_{-x^2+1}^{x^2+1} xydydx$$

(2)

$$\int_0^1 \int_0^1 ye^{xy}dydx$$

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(3)
$$\int_0^1 \int_y^{y^2} e^{x/y} dx dy$$

(4)
$$\iint_D (x^2 + y^2) dA$$
where D is the unit disk

(5)
$$\iint_D xy dA$$
where D is the unit disk

(6)
$$\iint_D x dA$$
where D is the region bounded by $x = -1$, $x = 1$, $y = 0$, and $y = 1 + \sqrt{1 - x^2}$.

(7)
$$\iint_D (x^2 + 2xy^2 + 2) dA$$
where D is the region bounded by $y = -x^2 + x$, the x axis, and the lines $x = 0$, $x = 2$.

(8)
$$\int_0^1 \int_0^1 \frac{dx dy}{\sqrt{1 + x + 2y}}$$

(9)
$$\int_1^4 \int_1^{\sqrt{x}} (x^2 + y^2) dy dx$$

(10)
$$\iint_D (x + y)^2 e^{x-y} dA$$
where D is the region bounded by $x + y = 1$, $x + y = 4$, $x - y = -1$, and $x - y = 1$.

(11)
$$\int_0^1 \int_{1-y}^1 (x + y^2) dx dy$$

(12)
$$\int_0^1 \int_0^z \int_0^y xy^2 z^3 dx dy dz$$

(13)
$$\int_0^1 \int_0^y \int_0^{x/\sqrt{3}} \frac{x}{x^2 + z^2} dz dx dy$$