MATH 1C PRACTICAL SPRING 2023 RECITATION 6

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1. Paths

A path is a continuous function $c : I \to \mathbb{R}^n$ for $I \subset \mathbb{R}$ an interval. (Draw a path). In this class, we consider paths to be piecewise C^1 or smooth.

The velocity of $c(t) = (c_1(t), \ldots, c_n(t))$ is its derivative $c'(t) = (c_1'(t), \ldots, c_n'(t))$. The velocity at any point is tangent to the path.

The arc length of $c : [a, b] \to \mathbb{R}^n$ is defined as

$$L(c) = \int_a^b \|c'(t)\| dt.$$

Exercise 1. Write a formula for a path which travels around the unit circle in $\mathbb{R}^2$ once counterclockwise. Calculate the arc length of the path.

2. Vector Fields

A vector field is a (C^1) function $F : \mathbb{R}^n \to \mathbb{R}^n$. For example, if $f : \mathbb{R}^n \to \mathbb{R}$ is a C^1 function, then its gradient $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous vector field.

Exercise 2. Let $f(x, y) = x^2 + y^2$. Sketch the gradient vector field ∇f together with some level sets of f.

Not every vector field is the gradient of a scalar-valued function. (Compare this to the fact that every continuous single-variable function has an antiderivative).

Exercise 3. Show that the vector field $V(x, y) = (y, -x)$ is not a gradient vector field (hint: equality of mixed partials).

If F is a vector field, a flow line or integral curve for F is a path $c(t)$ such that

$$c'(t) = F(c(t)).$$

That is, at every time t, the velocity of c is equal to the vector field at the point. An integral curve is the solution to a set of differential equations. By the Picard-Lindelof theorem from ODE theory, for some niceness conditions on F, given a starting point, there exists a unique integral curve going through the point for a small amount of time.

Exercise 4. Show that the curve $c(t) = (\sin t, \cos t, e^t)$ is a flow line of the vector field $F(x, y, z) = (y, -x, z)$.

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A differential operator takes functions in C^k for some k and produces functions which are some combination of the derivatives of the initial function. For example, the operator d/dx takes a function of one variable f and produces another function $\frac{df}{dx}$. Another example is the gradient, we can write

$$\nabla f = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right) f = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)$$

for any $f : \mathbb{R} \to \mathbb{R}^n$. In this sense, we can consider ∇ as an operator

$$\nabla = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right).$$

The divergence of a vector field $F = (F_1, F_2, F_3)$ is given by "taking the dot product of ∇ with F", i.e.

$$\operatorname{div} F = \nabla \cdot F = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3}.$$

The divergence takes a vector field and produces a scalar field. The physical interpretation of the divergence is that it represents the rate of expansion or compression of some fluid.

Exercise 5. Sketch and compute the divergence of the vector fields

(1) $F(x, y) = (x, y)$
(2) $G(x, y) = (-y, x)$.

We can take the divergence of the gradient of a function $f : \mathbb{R} \to \mathbb{R}^n$,

$$\begin{aligned} \nabla \cdot \nabla f &= \nabla \cdot \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right) \\ &= \frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2}. \end{aligned}$$

This is called the Laplacian operator $\Delta = \nabla \cdot \nabla$.

Another differential operator is the curl of a vector field $F : \mathbb{R}^3 \to \mathbb{R}^3$

$$\begin{aligned} \operatorname{curl} F = \nabla \times F &= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\ &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) i + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) j + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) k. \end{aligned}$$

Note that this is only meaningful for vector fields on $\mathbb{R}^3$.

Exercise 6. Show that for any C^2 function $f : \mathbb{R} \to \mathbb{R}^3$, that

$$\nabla \times (\nabla f) = 0,$$

that is, the curl of the gradient is the zero vector.