MATH 1C PRACTICAL SPRING 2023 RECITATION 2

ALAN DU

1. Optimization

In single variable calculus, to find the maxima or minima of a function $f : I \to \mathbb{R}$, for some interval I, first we find the critical points of f. If I is an open interval, those critical points must be the extrema. However, if I is closed, then we must also consider the value of f on the boundary points of I.

Let $f : U \to \mathbb{R}$ be a differentiable function, where $U \subset \mathbb{R}^n$. We want to find a point $x_0 \in U$ which maximizes or minimizes f on U. The way to solve this problem is the same as in the single variable case: we find the critical points, but we also need to consider if the extrema occur on the boundary of U. However, the boundary of a subset of $\mathbb{R}^n$ usually contains uncountably many points. How do we determine where the extrema occur on the boundary?

In this class, we generally consider subsets U which are defined using differentiable functions $\mathbb{R}^n \to \mathbb{R}$. For example, the closed unit ball in $\mathbb{R}^n$ is defined by

$$\overline{B_n} = \{x \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 \leq 1\}.$$

The boundary, which is the unit sphere, is the set of points

$$S_{n-1} = \{x \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 = 1\}.$$

In general, the region U is defined via a finite set of constraints $g_i(x) \leq c_i$, $g_i(x) < c_i$, $g_i(x) = c_i$, or $g_i(x) \neq c_i$ for some functions $g_i : \mathbb{R}^n \to \mathbb{R}$ and some constants c_i. For the constraints where the relation is $\leq$ or $=$, the boundary is defined by the constraints $g_i(x) = c_i$. If all the constraints are $<$ or $\neq$, then we say that U is open. If all the constraints are $\leq$ or $=$, then U is closed. For example, $\overline{B_n}$ is closed, but

$$B_n = \{x \in \mathbb{R}^n : \|x\| < 1\}$$

is open.

Finally, we say that U is bounded if there exists a constant C such that $\|x\| \leq C$ for all $x \in U$. Remark: we say that U is compact if it is closed and bounded.

Theorem 1.1 (First derivative test)**.** *Suppose U is open, if x_0 is an extremum of $f : U \to \mathbb{R}$, then $Df(x_0) = 0$.*

This is false of U is not open. For example, $f(x) = x$ on $U = [0, 1] \subset \mathbb{R}$ has minimum at $x = 0$, which is not a solution to $f'(x) = 0$. Also, the extrema of f may not exist. For example, $f(x) = x^3$ on $U = (-1, 1)$ has no minimum or maximum, even though it has a critical point at $x = 0$.

Date: April 27, 2023.

Again, if U is not open, then we must consider f on the boundary of U.

Theorem 1.2 (Lagrange multipliers)**.** *Let $g_1, \dots, g_m : \mathbb{R}^n \to \mathbb{R}$ be differentiable functions, and let $S = \{x : g_1(x) = \cdots = g_m(x) = 0\}$. If $x_0 \in S$ is a local extremum for f on S, and $\nabla g_1(x_0), \dots, \nabla g_m(x_0)$ are linearly independent, then there exist $\lambda_1, \dots, \lambda_m \in \mathbb{R}$ such that*

$$\nabla f(x_0) = \lambda_1 \nabla g_1(x_0) + \cdots + \lambda_m \nabla g_m(x_0).$$

In the case where we have one constraint $S = \{x : g(x) = 0\}$ such that $\nabla g \neq 0$ on S, then we can look for extrema by solving the system of equations

$$\begin{aligned} g(x) &= 0 \\ \nabla f(x) &= \lambda \nabla g(x) \end{aligned}$$

for x and λ.

Intuition for Lagrange multipliers: level sets of function are tangent to constraints.

Theorem 1.3. *Suppose U is closed and bounded, if f is continuous on U, then f achieves absolute extrema on U.*

Example 1. Determine whether $f(x, y) = x + y$ has absolute extrema on $U = \{(x, y) : x^2 + y^2 \leq 1\}$ and if so, find them.

Solution: draw the set U and level sets of f. Intuitively, the level curves of f, which in this case are straight lines, must be tangent to U.

Since U is closed and bounded, f achieves absolute extrema on U. We see that $Df = (1, 1) \neq 0$, hence there are no solutions in the interior of U by the first derivative test. Thus the extrema belong to the boundary $C = \{(x, y) : x^2 + y^2 = 1\}$.

Let $g(x, y) = x^2 + y^2 - 1$, then $\nabla g(x, y) = (2x, 2y) \neq 0$ on C. By Lagrange multipliers, if (x_0, y_0) is an extremum, then there exists λ such that

$$\begin{aligned} \nabla f(x_0, y_0) &= \lambda \nabla g(x_0, y_0), \\ (1, 1) &= \lambda(2x_0, 2y_0). \end{aligned}$$

This gives us $x_0 = y_0 = \frac{1}{2\lambda}$. Furthermore, $x_0^2 + y_0^2 = 1$. Hence $\lambda = \pm\frac{1}{\sqrt{2}}$.

If $\lambda = \frac{1}{\sqrt{2}}$, then $(x_0, y_0) = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ and $f(x_0, y_0) = \sqrt{2}$.

If $\lambda = -\frac{1}{\sqrt{2}}$, then $(x_0, y_0) = (-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$ and $f(x_0, y_0) = -\sqrt{2}$.

Comparing them, we see that $\sqrt{2}$ is the maximum and $-\sqrt{2}$ is the minimum.

Example 2. Let $f(x, y) = x^2$ and $U = \mathbb{R}^2$, then f achieves its minimum of 0 on the entire x-axis. Thus, the extrema of a function need not be an isolated set of points.

Exercise 1. Find the absolute extrema of the following functions in the given domains:

(1) $f(x, y, z) = x - y + z$, subject to $x^2 + y^2 + z^2 = 2$.
(2) $f(x, y) = x$, subject to $x^2 + 2y^2 \leq 3$
(3) $f(x, y) = x^2 + xy + y^2$ on the closed unit disk $D = \{(x, y) : x^2 + y^2 \leq 1\}$.

Exercise 2. Consider all rectangles with fixed perimeter p. Use Lagrange multipliers to show that the rectangle with maximal area is a square. (If you like, you can also do the problem without Lagrange multipliers).

2. Implicit Function Theorem

Reference: Rudin Chapter 9.

The setup here is that f is a C^1 mapping of an open set $E \subset \mathbb{R}^{n+m}$ into $\mathbb{R}^n$. We write (x, y), to denote a vector in $\mathbb{R}^{n+m}$, where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. If $A = Df(a, b)$, then $A(h, k) = A_x h + A_y k$, where $A_x : \mathbb{R}^n \to \mathbb{R}^n$, $A_y : \mathbb{R}^m \to \mathbb{R}^n$.

Theorem 2.1. *Let $f(a, b) = 0$ for some point $(a, b) \in E$. If $A = Df(a, b)$ is such that A_x is invertible, then there exist open sets $U \subset \mathbb{R}^{n+m}$ and $W \subset \mathbb{R}^m$ with $(a, b) \in U$ and $b \in W$ such that:*

For every $y \in W$, there corresponds a unique x such that $(x, y) \in U$ and $f(x, y) = 0$.

If this x is defined to be $g(y)$, then g is a C^1 mapping of W into $\mathbb{R}^n$, $g(b) = a$, $f(g(y), y) = 0$, and $g'(b) = -(A_x)^{-1} A_y$.