MATH 1C PRACTICAL SPRING 2023 RECITATION 2

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1. Differentiation

Reference: Chapter 9 of Walter Rudin's *Principles of Mathematical Analysis*.

Recall that a single-variable function f is differentiable at x if the limit

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

exists. In other words, there exists a real number $f'(x)$ such that

$$\lim_{h\to 0} \frac{|f(x+h)-f(x)-f'(x)h|}{|h|} = 0.$$

We can regard $f'(x)$ as a linear function which takes a number h and gives $f'(x)h$, and this function is special because it approximates the behavior of $f(x+h)-f(x)$.

Definition 1.1. Let $E \subset \mathbb{R}^n$ be open, $f : E \to \mathbb{R}^m$, and $x \in E$. If there exists a linear map $A : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{h\to 0} \frac{\|f(x+h)-f(x)-A(h)\|}{\|h\|} = 0, \tag{1.2}$$

then we say that f is differentiable at x, and we write $Df_x = f'(x) = A$. We call Df_x the (total) derivative or the differential of f at x.

The derivative is unique if it exists.

Theorem 1.3. *Suppose f is differentiable at x, and equation (1.2) holds with $A = A_1$ and $A = A_2$. Then $A_1 = A_2$.*

Proposition 1.4. *Equation (1.2) can be written in the form*

$$f(x+h)-f(x) = f'(x)h + r(h)$$

for some remainder $r(h)$ satisfying

$$\lim_{h\to 0} \frac{\|r(h)\|}{\|h\|} = 0.$$

Thus f is continuous at any point which f is differentiable.

The derivative satisfies the usual chain rule, but we have composition of linear maps instead of multiplication of real numbers.

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Theorem 1.5. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$, $g : \mathbb{R}^m \to \mathbb{R}^k$, and suppose f is differentiable at x_0, g is differentiable at $f(x_0)$. Then the mapping $g \circ f$ is differentiable at x_0 and*

$$F'(x_0) = g'(f(x_0))f'(x_0).$$

Definition 1.6. Let $f : \mathbb{R}^n \to \mathbb{R}$, $\{e_1, \dots, e_n\}$, the standard basis of $\mathbb{R}^n$, Define the partial derivative of f with respect to x_i at the point x as

$$\frac{\partial f}{\partial x_i}(x) = \lim_{t \to 0} \frac{f(x + te_i) - f(x)}{t},$$

provided the limit exists.

We can think of a function of a single variable $t \mapsto f(x + te_i)$, then the partial derivative is the derivative of this function at the point 0. We can also interpret the partial derivative as holding the other components of x constant and considering the single-variable derivative of f as we vary x_i.

Exercise 1. Let $f(x, y, z) = 2x + y^2x$, compute the partial derivatives of f.

For a function $f : \mathbb{R}^n \to \mathbb{R}^m$, we can consider the partial derivatives of the components of $f = (f_1, \dots, f_m)$, $\frac{\partial f_i}{\partial f_j}$.

Theorem 1.7. *Suppose f is differentiable at a point x, then the partial derivatives $\frac{\partial f_i}{\partial f_j}(x)$ exist, and*

$$f'(x)e_j = \left(\frac{\partial f_1}{\partial x_j}(x), \dots, \frac{\partial f_m}{\partial x_j}(x) \right).$$

Definition 1.8. As a consequence of the previous theorem, the matrix form of $f'(x)$ with respect to the standard bases is given by the $m \times n$ matrix

$$[f'(x)] = \text{Jac}(f)(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1}(x) & \cdots & \frac{\partial f_m}{\partial x_n}(x) \end{pmatrix}.$$

This is called the Jacobian of f at x. The Jacobian may exist even when the function is not differentiable.

Exercise 2. Use the chain rule to find $(f \circ g)'(-2, 1)$ for $f(u, v, w) = (v^2 + uw, u^2 + w^2, u^2v - w^3)$, $g(x, y) = (xy^3, x^2 - y^2, 3x + 5y)$.

In the case $m = 1$, the Jacobian is a $1 \times n$ vector usually called the gradient of f, denoted ∇f.

Definition 1.9. Suppose all the partial derivatives of f at x exist, and let $v \in \mathbb{R}^n$, then the directional derivative of f in the direction v is

$$\text{Jac}(f)(x)v.$$

In the case $m = 1$, this is just $\nabla f(x) \cdot v$.

Exercise 3. Find a direction in which the directional derivative of

$$f(x,y) = \frac{x^2 - y^2}{x^2 + y^2}$$

at $(1,1)$ is equal to 0.

Theorem 1.10. *If all the partial derivatives* $\frac{\partial f_i}{\partial x_j}(x)$ *of* f *exist and are continuous at* x*, then* f *is differentiable (in fact, continuously differentiable) at* x.

Exercise 4. Let $f(0,0) = 0$, and

$$f(x,y) = \frac{xy}{x^2 + y^2} \quad \text{if } (x,y) \neq (0,0).$$

Prove that the partial derivatives of f exist at every point of $\mathbb{R}^2$, but f is not continuous at $(0,0)$.

Definition 1.11. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be continuously differentiable, and let S be the surface consisting of those (x,y,z) satisfying $f(x,y,z) = k$ for k a constant. If $\nabla f(x_0,y_0,z_0) \neq 0$ for $(x_0,y_0,z_0) \in S$, the tangent plane of S at (x_0,y_0,z_0) is defined by the equation

$$\nabla f(x_0,y_0,z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0.$$