MATH 1C PRACTICAL SPRING 2023 RECITATION 1

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1. Euclidean Space

Definition 1.1. The standard inner product on $\mathbb{R}^n$ is defined as

$$x \cdot x = \langle x, y \rangle = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

for any $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$.

The norm of any vector in $\mathbb{R}^n$ is defined as

$$\|x\| = \sqrt{x \cdot x} = \sqrt{x_1^2 + \cdots + x_n^2}.$$

The distance between any two vectors x, y is defined as

$$\|x - y\|.$$

Exercise 1. Prove the parallelogram identity

$$\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2)$$

for any $u, v \in \mathbb{R}^n$.

In this course, we often restrict our attention to 2 or 3-dimensional Euclidean space.

The equation of a line in $\mathbb{R}^3$ is given by a function l of one parameter t of the form

$$l(t) = p + tv,$$

where $p, v \in \mathbb{R}^3$. The line passes through the point p and points in the direction v. In terms of the standard coordinates, we have

$$x = x_1 + at,$$
$$y = y_1 + bt,$$
$$z = z_1 + ct.$$

The equation of a plane in $\mathbb{R}^3$ which passes through (x_0, y_0, z_0) and has a normal vector $n = (A, B, C)$ is given by

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0,$$

or

$$Ax + By + Cz + D = 0.$$

Exercise 2. (1) Find the equation of the line passing through $(2, 1, 1)$ and $(0, 1, 0)$.
(2) Find an equation for the plane perpendicular to the vector $(-1, 1, -1)$ and passing through the point $(1, 1, 1)$.

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(3) Find an equation for the plane containing the points $(2, 1, -1), (3, 0, 2)$, and $(4, -3, 1)$
(4) Find an equation for a line that is parallel to the plane $2x - 3y + 5z - 10 = 0$ and passes through the point $(-1, 7, 4)$.

2. Real-Valued Functions of Multiple Variables

A function $\mathbb{R}^n \to \mathbb{R}$ is described in terms of the coordinates of $\mathbb{R}^n$. One example of a function $\mathbb{R}^2 \to \mathbb{R}$ is $f(x, y) = x^2 - y^2$.

A function $\mathbb{R}^n \to \mathbb{R}^m$ is described by an m-tuple of functions $\mathbb{R}^n \to \mathbb{R}$.

The graph of a function $f : \mathbb{R}^n \to \mathbb{R}$ is the set of all points $(x_1, \ldots, x_n, f(x_1, \ldots, x_n))$. In the case $n = 2$, we can visualize the graph as a subset of $\mathbb{R}^3$.

Another way to visualize functions is using level sets. The level set of a function $f : \mathbb{R}^n \to \mathbb{R}$ is the set of all points $(x_1, \ldots, x_n) \in \mathbb{R}^n$ such that $f(x_1, \ldots, x_n) = c$ for some fixed constant c.

Draw level sets and graph of $f(x, y) = x^2 - y^2$.

Exercise 3. Sketch the level curves and graphs of the following functions

(1) $f : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto x - y + 2$
(2) $f : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto x^2 + 4y^2$
(3) $f : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto -xy$.

Definition 2.1. Let $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ and let x_0 be in A or be a boundary point of A. Then we say $\lim_{x \to x_0} f(x) = b$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for any $x \in A$ satisfying $0 < \|x - x_0\| < \delta$, we have $\|f(x) - b\| < \epsilon$.

An easy way to show that a limit does not exist is to consider the limit along two different paths that approach the limiting point x_0. If two different values for the limit are found, then the limit does not exist.

Exercise 4. Compute the following limits or show they do not exist.

(1)
$$\lim_{(x,y)\to(0,0)} \frac{x^3 - y^3}{x^2 + y^2}$$
(2)
$$\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{y}$$
(3)
$$\lim_{(x,y)\to(0,0)} \frac{x^2}{(x^2 + y^2)}$$
(4)
$$\lim_{(x,y)\to(0,0)} \frac{\sin(2x) - 2x + y}{x + y}$$