MATH 1B ANALYTICAL W23 RECITATION 9

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1. Inner Product Spaces

Definition 1.1. Let $F = \mathbb{R}$ or $\mathbb{C}$, and let V be a vector space over F, then an inner product on V is a function $(\cdot, \cdot) : V \times V \to F$ satisfying for all $x, y, z \in V$, $\alpha, \beta \in F$,

- Linearity in the first argument: $(\alpha x + \beta y, z) = \alpha(x, z) + \beta(y, z)$,
- Conjugate symmetry: $(x, y) = \overline{(y, x)}$,
- Positive definiteness: $(x, x) \geq 0$ for all x, and $(x, x) = 0$ if and only if $x = 0$.

A space V together with an inner product is called an inner product space $(V, (\cdot, \cdot))$.

Given an inner product space $(V, (\cdot, \cdot))$, the norm is the function $\| \cdot \| : V \to \mathbb{R}$ defined by $\|x\| = \sqrt{(x, x)}$.

Theorem 1.2. *(Cauchy-Schwarz inequality)*

$$|(x, y)| \leq \|x\|\|y\|.$$

Lemma 1.3. *(Triangle inequality)*

$$\|x + y\| \leq \|x\| + \|y\|.$$

Lemma 1.4. *(Parallelogram identity)*

$$\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2).$$

Definition 1.5. In general, a norm on a vector space V is a function $\| \cdot \| : V \to \mathbb{R}$ satisfying for all $u, v \in V$, $\alpha \in F$,

- Homogeneity: $\|\alpha v\| = |\alpha|\|v\|$
- Triangle inequality: $\|u + v\| \leq \|u\| + \|v\|$
- Positive definiteness: $\|v\| \geq 0$ and $\|v\| = 0$ if and only if $v = 0$.

Not every norm on a vector space comes from an inner product.

2. Orthogonality

In this section we let $(V, (\cdot, \cdot))$ be an inner product space.

Definition 2.1. Two vectors $u, v \in V$ are orthogonal if $(u, v) = 0$. We write $u \perp v$.

We say v is orthogonal to a subspace $E \subset V$ if v is orthogonal to all vectors $w \in E$. We say subspaces E and F are orthogonal if all vectors in E are orthogonal to F.

To check two subspaces are orthogonal, it suffices to check orthogonality of the basis vectors.

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Proposition 2.2. *If u, v are orthogonal, then the Pythagorean identity holds:*

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2.$$

Definition 2.3. A set of vectors $\{v_1, \dots, v_n\}$ is called orthogonal if $(v_j, v_k) = 0$ for all $j \neq k$.

If, in addition $\|v_j\| = 0$ for all j, we call the system orthonormal.

Lemma 2.4. *If $\{v_1, \dots, v_n\}$ are orthonormal, then*

$$\left\| \sum_{k=1}^{n} a_k v_k \right\|^2 = \sum_{k=1}^{n} a_k^2.$$

Corollary 2.5. *Any orthogonal set of nonzero vectors is linearly independent.*

Definition 2.6. An orthogonal (orthonormal) system $\{v_1, \dots, v_n\}$ which is also a basis of V is called and orthogonal (orthonormal) basis.

Proposition 2.7. *Let $\{v_1, \dots, v_n\}$ be an orthonormal basis, then for any $x = \sum_{k=1}^{n} a_k v_k$ in V, we have $a_k = (x, v_k)$. Thus*

$$x = \sum_{k=1}^{n} (x, v_k) v_k.$$

Definition 2.8. Let E be a subspace of V, we define the orthogonal projection $P_E v$ of a vector v onto E as the vector $w \in E$ such that $v - w \perp E$. w exists and is unique.

Clearly $v \in E$ if and only if $P_E v = v$.

Theorem 2.9. *The orthogonal projection $w = P_E v$ minimizes the distance from v to E, i.e. for all $x \in E$, $\|v - w\| \leq \|v - x\|$, and $\|v - w\| = \|v - x\|$ if and only if $x = w$.*

Proposition 2.10. *Let $v_1, \dots, v_r$ be an orthogonal basis in E, then the orthogonal projection $P_E v$ of a vector v is given by*

$$P_E v = \sum_{k=1}^{r} \frac{(v, v_k)}{\|v_k\|^2} v_k.$$

The Gram-Schmidt process is used to convert a linearly independent system $x_1, \dots, x_n$ into an orthogonal system with the same span. We start by setting $v_1 = x_1$ and letting $E_1 = \mathrm{Span}(x_1)$.

Inductively, suppose for some $r < n$ that we have orthogonal vectors $v_1, \dots, v_r$ such that $\mathrm{Span}(x_1, \dots, x_r) = \mathrm{Span}(v_1, \dots, v_r) = E_r$. Define

$$v_{r+1} = x_{r+1} - P_{E_r} x_{r+1}.$$

Then since $x_{r+1} \notin E_r$ since the set is linearly independent, we have $v_{r+1} \neq 0$. It is clear that $\mathrm{Span}(v_1, \dots, v_{r+1}) = \mathrm{Span}(x_1, \dots, x_{r+1})$. Also v_{r+1} is orthogonal to v_k for all $k \leq r$.

Continue using this inductive process until we have pairwise orthogonal vectors $v_1, \dots, v_n$ with $\mathrm{Span}(x_1, \dots, x_n) = \mathrm{Span}(v_1, \dots, v_n)$.

We may choose to normalize the vectors $v_1, \dots, v_n$, i.e. replace each v_i by $v_i / \|v_i\|$, to get an orthonormal system.

Theorem 2.11. *Any finite-dimensional inner product space V has an orthonormal basis.*

Proof. Start with any basis of V and apply the Gram-Schmidt process. $\square$