MATH 1B ANALYTICAL W23 RECITATION 8

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Let V be a finite dimensional vector space over F, and $T : V \to V$ a linear map. Let $P(\lambda) = \det(A - \lambda I)$ be the characteristic polynomial of T, and let α be a root of P in F. Then α is an eigenvalue of T and has a nontrivial eigenspace.

Definition 0.1. We say the algebraic multiplicity of α is the multiplicity of α as a root of P, that is, the largest natural number k such that $(X - \alpha)^k$ divides $P(X)$.

We say the geometric multiplicity of α is the dimension of the eigenspace of α corresponding to T.

1. Diagonalization

Suppose there exists a basis for V consisting of eigenvectors $v_1, \ldots, v_n \in V$ for T, with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n \in F$. Then $Tv_i = \lambda_i v_i$, so relative to the basis $\{v_1, \ldots, v_n\}$, the matrix for T is diagonal $\operatorname{diag}(\lambda_1, \ldots, \lambda_n)$. In this case, we say that T is diagonalizable over F.

Because the matrix for T with respect to the eigenbasis is easy to work with, we might consider changing whatever basis we start with to the eigenbasis. There is a general procedure for changing bases.

Suppose we have two bases A, B for V, so relative to the basis A (for both the domain and the codomain), T is represented by the matrix $[T]_{AA}$, and relative to the basis B, T is represented by $[T]_{BB}$. We consider the change of basis matrix C from the basis A to the basis B, then $C = [I]_{BA}$ is just the matrix of the identity transformation, but we use A for the domain and B for the codomain. Then C is invertible and $C^{-1} = [I]_{AB}$ is the change of basis matrix from B to A. Hence, we can write

$$[T]_{AA} = [I]_{AB}[T]_{BB}[I]_{BA} = C^{-1}[T]_{BB}C.$$

If we express the basis vectors w_j of B in terms of the basis vectors v_i of A using

$$w_j = \sum_{i=1}^{n} a_{i,j} v_i,$$

then the change of basis matrix $C^{-1} = [I]_{AB}$ is the matrix $[a_{i,j}]$.

Definition 1.1. Two $n \times n$ matrices A and B are similar if there exists an invertible matrix C such that $A = C^{-1}BC$.

The relation of being similar is an equivalence relation. If A and B are similar, then they can be said to represent essentially the same linear transformation, but with respect to different bases.

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Theorem 1.2. *A matrix A (with values in F) is similar to a diagonal matrix D if and only if there exists a basis in F^n of eigenvectors of A, i.e. A is diagonalizable over F.*

Proof. Suppose $A = C^{-1}DC$ for $D = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$. Let $b_1, \dots, b_n$ be the columns of C^{-1}, and $B = \{b_1, \dots, b_n\}$ be the basis of these column vectors. Then $C^{-1} = [I]_{SB}$, for S the standard basis, so

$$D = CAC^{-1} = [I]_{BS}A[I]_{SB},$$

which means $D = [A]_{BB}$. Then clearly

$$[A]_{BB}b_k = \lambda_k b_k,$$

so $B = \{b_1, \dots, b_n\}$ is a basis of eigenvectors with corresponding eigenvalues $\lambda_1, \dots, \lambda_k$.

Conversely, if A has a basis of eigenvectors B, then letting $C = [I]_{BS}$ gives $A = C^{-1}DC$ as discussed above. □

Exercise 2: let A and B be similar $n \times n$ matrices. Let $A = C^{-1}BC$.

(1) By a homework problem, we know $\operatorname{Rank}(A) = \operatorname{Rank}(C^{-1}BC) \leq \operatorname{Rank}(B)$. Since $B = CAC^{-1}$, we know $\operatorname{Rank}(B) = \operatorname{Rank}(CAC^{-1}) \leq \operatorname{Rank}(A)$. Thus $\operatorname{Rank}(A) = \operatorname{Rank}(B)$.
(2) Follows from part 1 by the rank-nullity theorem.
(3) Suppose $Av = \lambda v$ (for λ in an algebraic closure of F), then

$$BCv = CAC^{-1}Cv = CAv = C\lambda v = \lambda Cv,$$

thus Cv is an eigenvector of B with eigenvalue λ. Similarly every eigenvalue of B is an eigenvalue of A, so A and B have the same eigenvalues.
(4) The characteristic polynomial of A is

$$\begin{aligned}\det(A - \lambda I) &= \det(C^{-1}BC - \lambda I) \\ &= \det(C^{-1}(B - \lambda I)C) \\ &= \det(C^{-1})\det(B - \lambda I)\det(C) \\ &= \det(B - \lambda I)\end{aligned}$$

is equal to the characteristic polynomial of B.

Note this also shows that the algebraic multiplicities of eigenvalues of A and B coincide.
(5) Trace is just the sum of the eigenvalues with algebraic multiplicities. Thus, this follows from part 4. Alternatively, since $\operatorname{tr}(MN) = \operatorname{tr}(NM)$ for any matrices M, N where the products make sense, we have

$$\operatorname{tr}(A) = \operatorname{tr}(C^{-1}BC) = \operatorname{tr}(BCC^{-1}) = \operatorname{tr}(B).$$

(6) We showed this in the proof of Theorem 1.2.

Exercise 3:

(a) Let $\{v_1, \dots, v_k\}$ be a basis of the eigenspace of λ_0, then extend this to a basis $B = \{v_1, \dots, v_k, \dots, v_n\}$ of $\mathbb{R}^n$.

(b) We calculate AP by each column of P: for the first k columns, we have the vectors v_i, and $Av_i = \lambda_0 v_i$. Hence, relative to the basis B, the first k columns of AP have λ_0 on the diagonal and 0 elsewhere. So if we use P to change from the basis B to the standard basis, we get

$$AP = P\begin{pmatrix} \lambda_0 I_k & X \\ 0 & Y \end{pmatrix}.$$

(c) Multiply the above equation by P^{-1} on both sides to obtain $A = PCP^{-1}$, i.e. A is similar to C.

(d) The characteristic polynomial of C is

$$\begin{aligned} \det(C - \lambda I) &= \det\begin{pmatrix} (\lambda_0 - \lambda)I_k & X \\ 0 & Y - \lambda I_{n-k} \end{pmatrix} \\ &= \det((\lambda_0 - \lambda)I_k)\det(Y - \lambda I_{n-k}) \\ &= (-1)^k(\lambda - \lambda_0)^k \det(Y - \lambda I_{n-k}). \end{aligned}$$

Thus the characteristic polynomial of C contains the factor $(\lambda - \lambda_0)$ at least k times.

(e) Since the characteristic polynomial of A equals the characteristic polynomial of C, the algebraic multiplicity of λ_0 of A is at least k.