MATH 1B ANALYTICAL W23 RECITATION 6

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1. Eigenvalues and Eigenvectors

Definition 1.1. Let $T : V \to V$ be a linear map, then an eigenvector of T is a vector $v \neq 0 \in V$ such that $Tv = \lambda v$ for some $\lambda \in F$ (remark: λ can be equal to 0). The scalar λ is called the eigenvalue of T corresponding to v.

To find the eigenvectors of T, we solve the equation

$$\det(A - \lambda I) = 0,$$

where A is the matrix of T. This gives a polynomial in the variable λ with coefficients in F, called the characteristic polynomial of T, and it is independent of the choice of bases for the matrix A. The eigenvalues are the roots of the characteristic polynomial.

After finding the eigenvalues, we can find the eigenvectors by solving the equation $Tv = \lambda v$ for each eigenvalue λ. For a fixed eigenvalue λ, the set of all eigenvectors of T with eigenvalue λ is called the corresponding eigenspace.

Suppose there exists a basis for V consisting of eigenvectors $v_1, \ldots, v_n \in V$ for T, with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n \in F$. Then $Tv_i = \lambda_i v_i$, so relative to the basis $\{v_1, \ldots, v_n\}$, the matrix for T is diagonal $\mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. In this case, we say that T is diagonalizable over F.

Theorem 1.2. *A matrix A (with values in F) admits a representation $A = S^{-1}DS$ where S is invertible and D is a diagonal matrix (both with values in F) if and only if there exists a basis in F^n of eigenvectors of A, i.e. A is diagonalizable over F.*

The proof is essentially that S is the change-of-basis matrix from the original basis for A to the basis of eigenvectors.

Exercise 1, 2, 3.

Considering the case $F = \mathbb{R}$, sometimes the polynomial $\det(A - \lambda I)$ may have complex roots, or even worse, it may have no real roots at all. For this reason, it is natural to allow eigenvalues to be complex numbers, then for A an $n \times n$ matrix, A will have exactly n complex eigenvalues counted with multiplicity. We can find eigenvectors for complex eigenvalues, but these eigenvectors will themselves have complex coefficients. Hence, there are no real eigenvectors corresponding to a non-real eigenvalue, a non-real eigenvalue still has complex eigenvectors. Therefore, it is important to specify which field we are working in; a matrix which is diagonalizable over $\mathbb{C}$ may not be diagonalizable over $\mathbb{R}$.

Exercise 4.

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