MATH 1B ANALYTICAL W23 RECITATION 6

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1. Determinant

The determinant is a function which takes in an $n \times n$ square matrix A with coefficients in F and outputs a scalar in F. It is in some sense a measure of volume.

The determinant has the following properties:

- It is linear in each column (row). That is,

$$\det(v_1, \dots, \alpha v_k + \beta u_k, \dots, v_n) = \alpha \det(v_1, \dots, v_k, \dots, v_n) + \beta \det(v_1, \dots, u_k, \dots, v_n).$$

- It is alternating, that is whenever two columns (rows) are identical, the determinant is 0.

$$\det(v_1, \dots, v, \dots, v, \dots, v_n) = 0.$$

- For I the identity matrix,

$$\det(I) = 1.$$

In fact, the determinant is the unique function on $n \times n$ matrices in F which has the above properties.

Recall the three types of elementary row operations we can do on matrices:

(1) Interchange row i with row j,
(2) Multiply row i by a non-zero scalar λ,
(3) Replace row i by its sum with λ times a multiple of row j.

The above properties imply that

Proposition 1.1. (1) *A row operation of type (1) multiplies the determinant by* -1.
(2) *A row operation of type (2) multiplies the determinant by* λ.
(3) *A row operation of type (3) does not change the determinant.*

Furthermore, if the rows form a linearly dependent set, then the determinant is 0*. In particular, if some row is entirely zeros, then the determinant is* 0*. Similar properties hold for columns.*

Proof. (1) Let our original matrix be $(v_1, \dots, v_i, \dots, v_j, \dots, v_n)$, and suppose we swap columns i and j. By linearity,

$$\det(v_1, \dots, v_j, \dots, v_i, \dots, v_n) + \det(v_1, \dots, v_j, \dots, v_j, \dots, v_n) \\ = \det(v_1, \dots, v_j, \dots, v_i + v_j, \dots, v_n).$$

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By the alternating condition, $\det(v_1, \ldots, v_j, \ldots, v_j, \ldots, v_n) = 0$, so

$$\det(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_n) = \det(v_1, \ldots, v_j, \ldots, v_i + v_j, \ldots, v_n).$$

By linearity,

$$\begin{aligned}\det(v_1, \ldots, v_j, \ldots, v_i + v_j, \ldots, v_n) &+ \det(v_1, \ldots, v_i, \ldots, v_i + v_j, \ldots, v_n) \\ &= \det(v_1, \ldots, v_i + v_j, \ldots, v_i + v_j, \ldots, v_n) = 0,\end{aligned}$$

so

$$\det(v_1, \ldots, v_j, \ldots, v_i + v_j, \ldots, v_n) = -\det(v_1, \ldots, v_i, \ldots, v_i + v_j, \ldots, v_n).$$

By linearity,

$$\begin{aligned}\det(v_1, \ldots, v_i, \ldots, v_i, \ldots, v_n) &+ \det(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_n) \\ &= \det(v_1, \ldots, v_i, \ldots, v_i + v_j, \ldots, v_n),\end{aligned}$$

so by the alternating condition

$$\det(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_n) = \det(v_1, \ldots, v_i, \ldots, v_i + v_j, \ldots, v_n).$$

Putting this all together gives us

$$\det(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_n) = -\det(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_n).$$

(2) This just follows directly from linearity.
(3) By linearity,

$$\begin{aligned}\det(v_1, \ldots, v_i + \lambda v_j, \ldots, v_n) &= \det(v_1, \ldots, v_i, \ldots, v_n) + \lambda \det(v_1, \ldots, v_j, \ldots, v_n) \\ &= \det(v_1, \ldots, v_i, \ldots, v_n)\end{aligned}$$

since $(v_1, \ldots, v_j, \ldots, v_n)$ has two identical columns equal to v_j.

Let $(v_1, \ldots, 0, \ldots, v_n)$ have a column of all 0's, then by linearity

$$\det(v_1, \ldots, 0, \ldots, v_n) = 0 \cdot \det(v_1, \ldots, v, \ldots, v_n) = 0$$

for any column vector v.

If the rows are linearly dependent, then some row i can be written as a linear combination of the other rows. Thus we can apply row operation (3) to reduce row i to a row of all zeros, so the determinant is 0. □

We can compute the determinant using the above properties. Specifically, we apply row reduction to put the matrix in reduced echelon form, while keeping track of whenever we use a row operation of type (1) or (2). If the REF is the identity matrix, then the determinant is the product of the inverses of the λ's we used in type (2) operations, times a -1 for each type (1) operation. Otherwise, if the REF is not the identity matrix, then the determinant is 0.

Exercise 1, 2.

One explicit definition of the determinant involves permutations and their signs.

Definition 1.2. A permutation of a set S is a bijective function $\sigma : S \to S$.

We usually consider permutations of the set $\{1, 2, \ldots, n\}$ for some number n, in which case a permutation is just a rearrangement of the numbers $1, 2, \ldots, n$. Denote the set of all permutations of $\{1, 2, \ldots, n\}$ as S_n. We can compose two permutations σ, τ to get a new permutation $\sigma\tau = \sigma \circ \tau$ using function composition (we call this the product of σ and τ, look up the symmetric group if you are curious why).

A transposition is a permutation $\tau = (i\ j)$ which swaps two numbers i, j while leaving all the others fixed. It is a fact that every permutation σ of $\{1, 2, \ldots, n\}$ can be written as a product of finitely-many transpositions. For example, the permutation (1 2 3), which sends 1 to 2, 2 to 3, and 3 to 1, can be written as

$$(1\ 2\ 3) = (2\ 3)(1\ 3).$$

While such a representation of σ as a product of transpositions is not unique, it is a fact that every such representation involves an even number of transpositions or every representation involves an odd number of transpositions.

Definition 1.3. Let σ be a permutation and let $\sigma = \tau_1 \cdots \tau_k$ be a product of transpositions. Then the sign of σ is defined as $\operatorname{sgn}(\sigma) = (-1)^k$.

Definition 1.4. Let $A = [a_{i,j}]$ be an $n \times n$ matrix. The determinant of A is defined as

$$\det A = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{\sigma(1),1} a_{\sigma(2),2} \cdots a_{\sigma(n),n}.$$

Example 1. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\det A = ad - bc$.

Exercise 3, 4.

Proposition 1.5. *Let A, B be $n \times n$ matrices, then*

$$\det(AB) = \det(A)\det(B).$$

Also, for A^T the transpose of A,

$$\det(A^T) = \det(A).$$

Proposition 1.6. *An $n \times n$ matrix A is invertible if and only if $\det(A) \neq 0$. If A is invertible, then*

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$