MATH 1B ANALYTICAL W23 RECITATION 5

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1. Solving systems of linear equations

We would like to solve equations of the form $Ax = b$ where A is some $m \times n$ matrix, b is a vector with m entries, and x is a vector with n entries which we would like to find. In the case where $m = n$ and A is invertible, then we can apply A^{-1} on both sides of the equation to get

$$x = A^{-1}b.$$

How do we deal with the general case?

Recall the procedure for finding the inverse of a matrix A: we apply some row operations to reduce A to the identity matrix, while keeping track of what those row operations do to the identity matrix. The general principle for solving systems of linear equations works the same way. There are three types of elementary row operations we can do on matrices:

(1) Interchange row i with row j,
(2) Multiply row i by a non-zero scalar λ,
(3) Replace row i by its sum with λ times a multiple of row j.

Consider the augmented matrix of the system

$$M = \left(\begin{array}{ccc|c} a_{1,1} & \dots & a_{1,n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m,1} & \dots & a_{m,n} & b_m \end{array}\right).$$

Proposition 1.1. *For a system of linear equations $Ax = b$, with M the augmented matrix, applying any of the above three types of elementary row operations to M does not change the set of solutions $\{x : Ax = b\}$.*

The proof is essentially that these operations are reversible, meaning that if we apply an operation to get from the old system to a new one, we can apply an operation to get from the new system back to the old one. This coincides with the fact that these row operations are exactly the result of multiplying M by a special kind of invertible matrix, called an elementary matrix.

Definition 1.2. A matrix is in echelon form is it satisfies the following conditions:

- All zero rows, if any, are below all non-zero rows
- For any non-zero row, its leading entry (called the pivot) is strictly to the right of the leading entry in the previous row.

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A matrix is in reduced echelon form if it is in echelon form and

- All pivot entries are equal to 1
- All entries above the pivots are 0.

Proposition 1.3. *Every matrix can be reduced using elementary row operations to a matrix in reduced echelon form.*

Proof. Apply the following algorithm to the matrix

(1) Find the leftmost non-zero column of the matrix.
(2) Apply row operations of type (1) and type (2) to make the uppermost entry of this column equal to 1. This is the pivot.
(3) Add an appropriate multiple of the first row to each of the below rows to make all entries below the pivot equal to 0.
(4) Repeat on the matrix where we ignore the first row, i.e. after applying the first three steps, we leave the row unchanged in the rest of the process and work on the rows below it.

After finitely many loops of the algorithm, we have put the matrix in echelon form. To put it in reduced echelon form, we can kill the non-zero entries above the pivots by adding suitable multiples of the pivot row. □

Once the matrix is in reduced echelon form, it is easy to read off the solutions to the system of equations.

Proposition 1.4. *The system $Ax = b$ has a solution if and only if the reduced echelon form of its augmented matrix has no pivot in the last column, i.e. it has no row of the form*

$$\begin{pmatrix} 0 & 0 & \dots & 0 & | & c \end{pmatrix}$$

for $c \neq 0$.

Exercise 1 and 2.

Proposition 1.5. *Let A be the matrix of a linear transformation T, and let A_e be the echelon form of A. Then*

- *The pivot columns of the original matrix A give a basis of $\mathrm{Im}(T)$.*
- *A basis of $\ker(T)$ is given by solving the equation $Ax = 0$.*

2. Determinant

Gloss over the definition of determinant. Mention that determinants can be computed using block matrices.

Proposition 2.1. *Let A, B be $n \times n$ matrices, then*

$$\det(AB) = \det(A)\det(B).$$

Also, for A^T the transpose of A,

$$\det(A^T) = \det(A).$$

Proposition 2.2. *An $n \times n$ matrix A is invertible if and only if* $\det(A) \neq 0$. *If A is invertible, then*

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$