MATH 1B ANALYTICAL W23 RECITATION 4

ALAN DU

1. Invertible Transformations

Definition 1.1. Let $T : V \to W$ be a linear transformation, then we say T is invertible if there exists a linear transformation $T^{-1} : W \to V$ such that

$$T^{-1}T = \mathrm{id}_V, \quad TT^{-1} = \mathrm{id}_W .$$

If A is a matrix, we say A is invertible if there exists a matrix A^{-1} such that

$$A^{-1}A = I, \quad AA^{-1} = I.$$

In the case of the above definition, the map T^{-1} is unique and is called the inverse of T. The matrix A^{-1} is unique and is called the inverse of A. Note that T is invertible if and only if its matrix is invertible.

Proposition 1.2. *If $T : V \to W$ is invertible, then $\dim V = \dim W = n$. As a result, if A is an invertible matrix, then A must be a square matrix.*

Proposition 1.3. *If A and B are invertible matrices, then their product AB is invertible with inverse $(AB)^{-1} = B^{-1}A^{-1}$.*

Ask for examples of invertible linear transformations/matrices. E.g. the identity matrix, reflections, rotations in $\mathbb{R}^2$. Exercise 1.

Proposition 1.4. *For a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $ad - bc \neq 0$, the inverse is given by*

$$\frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Do an example of finding the inverse of a 3×3 matrix.

Inverse matrices can be used to solve systems of linear equations. Suppose we have a system of the form

$$\begin{aligned} a_{1,1}x_1 + \cdots + a_{1,n}x_n &= y_1, \\ &\vdots \\ a_{n,1}x_1 + \cdots + a_{n,n}x_n &= y_n, \end{aligned}$$

Date: January 26, 2023.

then we can write this in matrix form

$$\begin{pmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & \vdots & \vdots \\ a_{n,1} & \dots & a_{n,n} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix},$$

or more simply,

$$Ax = y.$$

Now if A is invertible, then we can multiply both sides by A^{-1} to obtain

$$x = A^{-1}y.$$

This gives us the solution x to the system of equations.

Exercise 2.

2. Rank-Nullity Theorem

Definition 2.1. Let $T : V \to W$ be a linear transformation. The kernel of T, denoted $\ker T$, is the subspace of vectors $v \in V$ such that $Tv = 0$. The dimension of $\ker T$ is called the nullity of T.

The image of T, denoted $T(V)$ or $\operatorname{Im} T$, is the subspace of vectors $w \in W$ such that $w = Tv$ for some $v \in V$. The dimension of $T(V)$ is called the rank of T.

Theorem 2.2. *Let $T : V \to W$ be a linear transformation, then*

$$\dim V = \dim \ker T + \dim T(V).$$

Exercise 3 and 4.