MATH 1B ANALYTICAL W23 RECITATION 3

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In the following, let V, W be vector spaces over the field F.

Definition 0.1. A transformation $T : V \to W$ is linear if for all $u, v \in V$ and $a \in F$,

- T(u+v)=T(u)+T(v)
- T(av)=aT(v).

Let V, W be vector spaces with bases $\{v_1, \dots, v_n\}$ and $\{w_1, \dots, w_m\}$, respectively. Let $T : V \to W$ be a linear transformation; we consider what T does to the basis vectors $\{v_1, \dots, v_n\}$. Write

$$T(v_i) = \sum_{j=1}^{m} a_{j,i} w_j,$$

then since T is linear, the value of T on any vector v is determined by these $m \times n$ numbers $a_{j,i} \in F$. In particular, if $v = \sum_{i=1}^{n} x_i v_i$, then

$$\begin{aligned} T\left(\sum_{i=1}^{n} x_i v_i\right) &= \sum_{i=1}^{n} x_i T(v_i) \\ &= \sum_{i=1}^{n} x_i \sum_{j=1}^{m} a_{j,i} w_j \\ &= \sum_{j=1}^{m} \sum_{i=1}^{n} a_{j,i} x_i w_j. \end{aligned}$$

Written in vector form, what we have is

$$T \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{n} a_{1,i} x_i \\ \sum_{i=1}^{n} a_{2,i} x_i \\ \vdots \\ \sum_{i=1}^{n} a_{m,i} x_i \end{pmatrix} = A \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix},$$

where the operation on the right is matrix multiplication with

$$A = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{pmatrix}.$$

What this calculation shows is that all the information of the linear transformation T is contained in the $m \times n$ matrix A.

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Definition 0.2. In the setting as above, we call the matrix $A = (a_{j,i})$ the matrix of the linear transformation $T : V \to W$ with respect to the bases $\{v_1, \ldots, v_n\}$ of V and $\{w_1, \ldots, w_m\}$ of W.

This matrix very much depends on the choice of bases for V and W.

Example 1. Let $C^0([0,1]) = \{\text{continuous real-valued functions on } [0,1]\}$, $C^1([0,1]) = \{\text{continuously-differentiable real-valued functions on } [0,1]\}$, then the differentiation map $\frac{d}{dx} : C^1([0,1]) \to C^0([0,1])$ is linear. If we restrict our attention to $\frac{d}{dx} : V \to V$, where V is the set of polynomials of degree at most n, then $\frac{d}{dx}x^m = mx^{m-1}$, so with respect to the basis $\{1, x, x^2, \ldots, x^n\}$, the matrix of $\frac{d}{dx}$ is equal to

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & n \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Proposition 0.3. *Let U, V, W be vector spaces, $S : U \to V$ and $T : V \to W$ be linear transformations, and A and B the matrices of S, T, respectively. Then the composition $T \circ S$ is a linear map from $U \to W$ which has matrix BA.*