MATH 1B ANALYTICAL W23 RECITATION 2

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In the following, let V be a vector space over the field F.

Definition 0.1. Let $v_1, \ldots, v_n \in V$ be a set of vectors in V, then a linear combination of these vectors is a sum

$$v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = \sum_{k=1}^{n} a_k v_k$$

where $a_1, \ldots, a_n \in F$. This sum gives a new vector $v \in V$.

Suppose we are given a set of vectors $v_1, \ldots, v_n$; we often consider solutions to equations of the form

$$v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$$

where v is an arbitrary vector in V and $a_1, \ldots, a_n$ are unknowns in F. In other words, we would like to write v as a linear combination of $v_1, \ldots, v_n$. For fixed v, two natural questions arise: does a solution to such a system exist, and if it does, is it unique? This motivates the following definitions.

Definition 0.2. The span of a set of vectors $v_1, \ldots, v_n \in V$ is the set of all linear combinations of $v_1, \ldots, v_n$, that is the set

$$\mathrm{Span}(v_1, \ldots, v_n) = \{a_1 v_1 + \cdots + a_n v_n \mid a_1, \ldots, a_n \in F\}.$$

The set $v_1, \ldots, v_n$ is said to span V, or be a spanning set of V, if $V = \mathrm{Span}(v_1, \ldots, v_n)$, i.e. any $v \in V$ can be written as a linear combination of $v_1, \ldots, v_n$, i.e. for any $v \in V$, there exist $a_1, \ldots, a_n \in F$ such that

$$v = a_1 v_1 + \cdots + a_n v_n.$$

Exercise 1. Show that the span of any set of vectors in V is a subspace of V.

Definition 0.3. A set of vectors $v_1, \ldots, v_n \in V$ is said to be linearly independent if the only linear combination of the vectors which equals 0 is the trivial combination, meaning the equation

$$a_1 v_1 + \cdots + a_n v_n = 0$$

has only the trivial solution $a_1 = \cdots = a_n = 0$. The set is linearly dependent if it is not linearly independent, i.e. there exist some $a_1, \ldots, a_n \in F$ not all 0 such that

$$a_1 v_1 + \cdots + a_n v_n = 0.$$

Proposition 0.4. *A set of vectors $v_1, \ldots, v_n \in V$ is linearly dependent if and only if one of the vectors v_k can be represented as a linear combination of the other vectors.*

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Definition 0.5. A set of vectors $v_1, \dots, v_n \in V$ is a basis for V if it is a linearly independent spanning set of V.

Assuming the axiom of choice, it is true that every vector space has a basis.

Proposition 0.6. *A set of vectors $v_1, \dots, v_n \in V$ is a basis for V if and only if any vector $v \in V$ has a unique representation as a linear combination*

$$v = a_1 v_1 + \cdots + a_n v_n.$$

Remark. The above definitions and propositions also work for an infinite set of vectors, with the caveat that all sums must be *finite* sums. That is, if $S \subset V$ is a set of vectors, then a linear combination of vectors in S is a sum

$$v = a_1 v_1 + \cdots + a_n v_n \ : \ a_1, \dots, a_n \in F, \ v_1, \dots, v_n \in S.$$

Theorem 0.7. *Let S, T be two bases for V, then S and T have the same cardinality. In particular, if $S = \{v_1, \dots, v_n\}$ is a finite basis for V, then any basis of V must be finite and have the same number of elements as S.*

Definition 0.8. In the case of the above theorem where $S = \{v_1, \dots, v_n\}$ is finite, we say that V is a finite dimensional vector space and the number n is called the dimension of V. If V does not have a finite basis, then V is called an infinite dimensional vector space.

Example 1. Take $V = \mathbb{R}^n$ as a vector space over $\mathbb{R}$, then the vectors $e_1, \dots, e_n$ where e_i has entries all 0 except for the entry in the i-th spot, which is 1, is a basis of $\mathbb{R}^n$, called the standard basis.

Example 2. Let $\mathbb{R}[X]$ be the set of all polynomials in the variable X having real coefficients, considered as a vector space over $\mathbb{R}$. Then $\mathbb{R}[X]$ is an infinite dimensional vector space: if $p_1, \dots, p_n$ is a finite set of polynomials, then let d be the maximum degree of any of the p_i. Clearly X^{d+1} is not a linear combination of $p_1, \dots, p_n$.

However, $\mathbb{R}[X]$ does have the infinite basis $\{1, X, X^2, \dots, X^n, \dots\}$.